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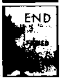

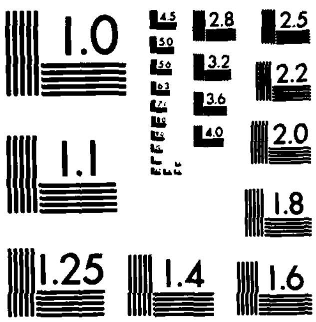
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Range-residuated mappings are introduced and their order theoretic and semigroup properties studied. Applications are given to cluster analysis as well as to the simplification of digital images.

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# RANGE RESIDUATED MAPPINGS

M. F. Janowitz*

1. Introduction. A digital picture may be thought of as a mapping $d:X \to L$ where $X$ is a finite set and $L$ a finite chain or the cartesian product of finitely many such chains. The idea is that $X$ is of the form $S \times T$, where $S$ is the set consisting of the first $s$, and $T$ the set consisting of the first $t$ positive integers, while $L$ represents the numerical coding of the brightness settings of the color guns that produce the picture. For a monochromatic picture, there would be only a single gun, so that $L$ would be a chain. Thus $d(x)$ yields the color or intensity level at site $x$. The mapping $d$ produces a clustering of $X$ into disjoint subsets by the rule

$$A_h = \{x \in X : d(x) = h\} \qquad (h \in L) \quad .$$

It is sometimes convenient to think instead of the clusters

$$B_h = \{x \in X : d(x) \leq h\} \qquad (h \in L)$$

and note that this produces a situation quite analogous to the model for cluster analysis that was described in [2]. In order to demonstrate an essential difference between the two situations, it turns out to be useful to examine in some detail the nature of the earlier model. One is given a finite (nonempty) set $X$ and a dissimilarity measure on $X$. This is a mapping $d:X \times X \to L$, where $L$ denotes the nonnegative reals and $d$ satisfies



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(DC1) $d(a,b) = d(b,a)$

(DC2) $d(a,a) = 0$

for all $a,b \in X$. One associates with $d$ a numerically stratified clustering $Td:L \to P(X \times X)$ defined by the rule

$$Td(h) = \{(a,b):d(a,b) \leq h\} \qquad (h \in L).$$

The mapping $Td:L \to P(X \times X)$ turns out to be residual in the sense of [1], p. 11. This situation may then be generalized by taking $L$ to be a join semilattice with $0$, replacing $P(X \times X)$ with a bounded poset $M$, and defining an $L$-stratified clustering to be a residual mapping $C:L \to M$ as in [2], p. 61. It is useful to recall here that $C:L \to M$ is residual if $C$ is isotone and there exists an isotone mapping $C^*:M \to L$ such that

(1) $C^*C(h) \leq h$

(2) $CC^*(m) \geq m$

for all $m \in M$, $h \in L$. The mapping $C^*$ is called the residuated mapping associated with $C$, and the reader is referred to [1] for further details. One often wishes to take a residual mapping $C:L \to M$ and shift the output levels by means of a mapping $\theta:L \to L$. The only reasonable choice for such a $\theta$ is to take $\theta$ to be residual since one is then guaranteed that $C \circ \theta:L \to M$ is residual. Now this treats the $0$ element of $L$ as a distinguished element, since $\theta^*(0) = 0$ for every residuated mapping $\theta^*$ on $L$. This makes sense in the cluster analysis context, since $d(a,b) = 0$ is generally taken to mean that $a,b$ cannot be distinguished in terms of the given input data.

In the context of digital images, one does not wish to distinguish the 0 element of $L$ in the above manner. In order to avoid this, it becomes necessary to modify the notion of an $L$-stratified clustering. Specifically, we shall drop the requirement that $M$ have a least element and consider mappings $C^*:M \to L$ that are residuated when considered as mappings from $M$ into the order filter generated by their range. Thus there exists an isotone mapping $C:F \to M$, where $F$ denotes the aforementioned order filter, and $C, C^*$ are linked by the requirement that

$$(3)\quad CC^*(m) \geq m \quad \text{for all} \quad m \in M$$

$$(4)\quad C^*C(h) \leq h \quad \text{provided} \quad h \geq \text{some} \quad C^*(m) \quad \text{for} \quad m \in M.$$

By [1], Theorem 2.5, p. 10, this amounts to saying that the preimage under $C^*$ of a principal ideal of $L$ is either empty or itself a principal ideal of $L$. To be more specific, if we are to work with a digital picture, we are given a finite nonempty set $X$ and a mapping $d:X \to L$. If $P'(X)$ denotes the semilattice formed by the nonempty subsets of $X$, then $d$ may be extended to a mapping $d^*:P'(X) \to L$ by the rule

$$(5)\quad d^*(A) = \vee\{d(x):x \in A\}$$

for every nonempty subset $A$ of $X$. It is then easy to see that $d^*$ is residuated on the order filter generated by its range. Such mappings will henceforth be called <u>range-residuated</u>. They have already been used in [3] in connection with an investigation of ordinal filters in digital imagery, and in [4] in connection with a characterization of the semilattice of weak orders on a finite set. We agree to let $RR(P,Q)$ denote the collection of range-residuated mappings of the poset $P$ into the poset $Q$, and

$RR^+(Q,P)$ the associated collection of residual mappings from order filters of $Q$ into $P$. In case $P = Q$, we shall use $RR(P)$ and $RR^+(P)$ in place of $RR(P,P)$ or $RR^+(P,P)$. If $P$ is a finite chain then $RR(P)$ is nothing more than the set of all isotone mappings on $P$, while if $P$ is a finite join semilattice, then $RR(P)$ conissts of the join endomorphisms of $P$. If digital pictures are thought of as elements $C$ of $RR^+(L,M)$, and if $L$ is a finite chain, this shows that the levels of $C$ may be shifted by means of any isotone mapping $\theta$ on $L$ to produce a new picture $C\theta \in RR^+((L,M)$. In view of all this, we now embark on an investigation into order theoretic properties of these mappings.

2. *Range-Residuated Mappings*. Let $P,Q$ be posets each having a largest element $1$. For each $q \in Q$, the constant mapping $\kappa_q : P \to Q$ defined by $\kappa_q(x) = q$ for all $x \in P$ is range-residuated, with $\kappa_q^+$ given by $\kappa_q^+(y) = 1_p$ for all $y \geq q$. If $Q$ happens to be a join semilattice, then the join translation $\tau_q(x) = x \vee q$ is in $RR(Q)$ with $\tau_q^+(y) = y$ for all $y \geq q$. Before proceeding, let us develop some elementary properties of range-residuated mappings. They are basically generalizations of results on residuated mappings, but are included here for completeness.

*THEOREM 1* (see [1], Theorem 2.8, p.14). *Let* $P,Q,S$ *be posets.* $\phi \in RR(P,Q)$ *and* $\psi \in RR(Q,S)$. *Then* $\psi\phi : P \to R$ *is range-residuated with* $(\psi\phi)^+ = \phi^+ \circ \psi^+$.

*Proof:* Evidently $\psi\phi : P \to R$ is isotone. If $p \in P$, then $\psi\phi(p)$ is in the domain of $\psi^+$, so that $\psi^+\psi\phi(p) \geq \phi(p)$ and we have

$\phi^+\psi^+\psi\phi(p) \geq \phi^+\phi(p) \geq p$. On the other hand, if $s \geq \psi\phi(p)$, then $\psi^+(s) \geq \phi(p)$ puts $\psi^+(s)$ in the domain of $\phi^+$. Thus $\phi^+\psi^+(s)$ can be formed and $\psi\phi\phi^+\psi^+ \leq \psi\psi^+(r) \leq r$. In that the domain of $\phi^+ \circ \psi^+$ is precisely the order filter generated by the range of $\psi\phi$, this completes the proof.

COROLLARY 2. RR(P) *forms a semigroup with identity.*

Proof: The identity map acts as a multiplicative identity element for RR(P).

Assuming that mappings are written on the left, we also have

COROLLARY 3. RR(P) *has a left (but not right) zero element.*

Proof: Let $x \in p$ and $\phi \in RR(P)$. One simply notes that

$$\phi\kappa_x = \kappa_{\phi(x)} \quad \text{and} \quad \kappa_x\phi = \kappa_x,$$

so that $\kappa_x$ is a left (but not right) zero element for RR(P).

It is easy to show that any left zero element of RR(P) is of the form $\kappa_x$ for some $x \in P$. Of special interest is the case where $P$ is bounded and one works with $\kappa_0$.

If $\phi: P \to Q$ is a residuated mapping with associated residual mapping $\phi^+: Q \to P$, and if both $P$ and $Q$ are equipped with their dual orderings, then $\phi^+$ becomes residuated with $\phi$ its associated residual mapping. This leads to an obvious duality between residuated and residual mappings. This duality does not carry over to range-residuated mappings since $\phi \in RR(P,Q)$

has an associated residual mapping whose domain is an order filter of $Q$ rather than being all of $Q$. Bearing this in mind, we agree to say (as in [4]) that $\phi \in RR(P,Q)$ is range-closed if $\phi(a) \leq q \leq \phi(p)$ implies $q \in \text{range } \phi$; to say that $\phi$ is dually range-closed will be to say that the range of $\phi^+$ is an order filter of $P$. An obvious modification of the proof of [1], Theorem 13.1, p. 119 now produces

THEOREM 4. *Let* $P,Q$ *be bounded posets. For* $\phi \in RR(P,Q)$, *the following are equivalent:*

(1) $\phi$ *is range-closed.*

(2) *The restriction of* $\phi$ *to* $[\phi^+\phi(0), 1]$ *is a surjection onto* $[\phi(0), \phi(1)]$.

(3) *In the interval* $[\phi(0), 1]$ *of* $Q$, $q \wedge \phi(1)$ *exists and equals* $\phi\phi^+(q)$.

(4) $\phi^+$ *is injective.*

Similarly, an obvious modification of the proof of [1], Theorem 13.1*, p. 119 would produce

THEOREM 5. *Let* $P,Q$ *be bounded posets. For* $\phi \in RR(P,Q)$, *the following are equivalent:*

(1) $\phi$ *is dually range-closed.*

(2) $\phi^+$ *is a surjection onto* $[\phi^+\phi(0), 1]$.

(3) *For all* $p \in P$, $p \vee \phi^+\phi(0)$ *exists and equals* $\phi^+\phi(p)$.

(4) *The restriction of* $\phi$ *to* $[\phi^+\phi(0), 1]$ *is injective.*

As in [1], p. 120, we also agree to call $\phi \in RR(P,Q)$ weakly regular in case $\phi$ is both range-closed and dually range-closed. Examples of such mappings are provided by the constant mappings $\kappa_x$ as well as by the join translations $\tau_x$. The analog of [1], Theorem 13.2, p. 121 may now be stated as

THEOREM 6. *Let $P,Q$ be bounded posets.*

(1) *If $\phi \in RR(P,Q)$ is weakly regular, then its restriction to $[\phi^+\phi(0), 1]$ is an isomorphism onto $[\phi(0), \phi(1)]$; furthermore, for $p \in P$ and $q \geq \phi(0)$, we have that $p \vee \phi^+\phi(0)$ exists and is given by $\phi^+\phi(p)$, and that $q \wedge \phi(1)$ exists in $[\phi(0), 1]$ and is given by $\phi^+\phi(q)$.*

(2) *Let $a \in P$ and $b,c \in Q$ with $b < c$. Suppose that $p \vee a$ exists for all $p \in P$, that $q \wedge c$ exists for all $q \geq b$ in $Q$, and that $\tau$ is an isomorphism of $[a,1]$ onto $[b,c]$. If $\phi:P \to Q$ is defined by $\phi(p) = \tau(p \vee a)$, then $\phi \in RR(P,Q)$, $\phi$ is weakly regular, and $\phi^+$ is given by $\phi^+(q) = \tau^{-1}(q \wedge c)$ for $q \geq b$.*

Recall now that a pair $(a,b)$ of elements of a lattice is modular and denoted $M(a,b)$ if $x \leq b$ implies that $x \vee (a \wedge b) = (x \vee a) \wedge b$; dually, a dual modular pair is denoted $M^*(a,b)$ and signifies that $x \geq b$ implies $x \wedge (a \vee b) = (x \wedge a) \vee b$. We then have

THEOREM 7. *Let $P$ be a bounded lattice and $\phi \in RR(P)$ a range-closed idempotent. Then $M(\phi^+\phi(0), \phi(1))$ holds.*

Proof: Let $a = \phi^+\phi(0)$ and $b = \phi(1)$. If $a \wedge b \leq x \leq b$, then $x = \phi(y)$ for some $y \geq a$ by Theorem 4. Hence

$$x = \phi\phi^{+}\phi^{+}\phi(x) \geq \phi\phi^{+}(x \vee a) = (x \vee a) \wedge b \geq x$$

shows $x = (x \vee a) \wedge b$. In general, if $x \leq b$, then $a \wedge b \leq x \vee (a \wedge b) \leq b$ shows that

$$x \vee (a \wedge b) = [x \vee (a \wedge b) \vee a] \wedge b = (x \vee a) \wedge b,$$

whence $M(a,b)$.

Dually, we have

<u>THEOREM 8</u>. <u>Let</u> $P$ <u>be a bounded lattice and</u> $\phi \in RR(P,Q)$ <u>a dual range-</u>closed idempotent. Then $M^*(\phi(1), \phi^{+}\phi(0))$, and $1 = \phi(1) \vee \phi^{+}\phi(0)$.

Combining Theorems 7 and 8, we generalize [1], Theorem 13.4, p. 123.

<u>THEOREM 9</u>. <u>Let</u> $P$ <u>be a lattice and</u> $\phi \in RR(P)$. <u>The following are necessary and sufficient conditions for</u> $\phi$ <u>to be a weakly regular idempotent</u>:

(1) $\phi^{+}\phi(0) \vee \phi(1) = 1$

(2) $M(\phi^{+}\phi(0), \phi(1))$ and $M^*(\phi(1), \phi^{+}\phi(0))$

(3) $\phi(x) = [x \vee \phi^{+}\phi(0)] \wedge \phi(1)$.

<u>Proof</u>: Let $a \vee b = 1$, $M(a,b)$ and $M^*(b,a)$. Define $\phi$ and $\psi$ by

$$\phi(x) = (x \vee a) \wedge b \qquad (x \in p)$$
$$\psi(x) = (x \wedge b) \vee a \qquad (x \geq a \wedge b).$$

Then

$$\psi\phi(x) = [(x \vee a) \wedge b] \vee a = x \vee a \geq x$$

and for $x \geq a \wedge b$,

$$\begin{aligned}\phi\psi(x) &= [(x \wedge b) \vee a] \wedge b \\ &= (x \wedge b) \vee (a \wedge b) = x \wedge b \leq x.\end{aligned}$$

Thus $\phi \in RR(P)$ with $\psi = \phi^+$. The fact that $\phi$ is a weakly regular idempotent is now also clear. For the converse, apply Theorems 7 and 8.

Continuing along these lines, we say that a range-residuated mapping $\phi \in RR(P,Q)$ is totally range-closed if the image under $\phi$ of a principal ideal of $P$ is necessarily a convex subset of $Q$. We then have

THEOREM 10 (See [1], Theorem 13.5, p. 124). Let $P$ be a bounded lattice. The following conditions on a element $\phi$ of $RR(P)$ are then equivalent:

(1) $\phi$ is totally range-closed.

(2) $\phi$ range-closed implies $\phi\psi$ range-closed for every $\psi \in RR(P)$.

(3) For $x \geq \phi(0)$, $y \in L$, $\phi[\phi^+(x) \wedge y] = x \wedge \phi(y)$.

Proof: (1) $\Longrightarrow$ (2) is clear.

(2) $\Longrightarrow$ (3) If $x \geq \phi(0)$, choose a residuated mapping $\psi$ on $P$ so that $\psi(1) = y$. Then $\phi\psi$ is range-closed, and we note that

$$\phi[\phi^+(x) \wedge y] = \phi\psi\psi^+\phi^+(x) = (\phi\psi)(\phi\psi)^+(x) = x \wedge \phi\psi(1) = x \wedge \phi(y).$$

The fact that $\psi(0) = 0$ was used to guarantee that $\psi^+\phi^+(x)$ could be formed.

(3) $\Longrightarrow$ (1) Let $b \in P$. We are to show that $\phi([0,b]) = [\phi(0), \phi(b)]$. But if $\phi(0) \leq x \leq \phi(b)$, then by (3),

$$x = \phi(b) \wedge x = \phi[b \wedge \phi^+(x)].$$

If we agree to call $\phi \in RR(P,Q)$ <u>dual totally range-closed</u> in case the image under $\phi^+$ of a principal filter of the domain of $\phi^+$ is a principal filter of $P$, we then have

<u>THEOREM 11. Let P be a bounded lattice, and</u> $\phi \in RR(P)$. <u>The following are then equivalent</u>:

(1) $\phi$ <u>is dual totally range-closed</u>.

(2) $\psi$ <u>dual range-closed implies</u> $\psi\phi$ <u>dual range-closed</u>.

(3) For $y \geq \phi(0)$, $x \in L$, $\phi^+[\phi(x) \vee y] = x \vee \phi^+(y)$.

The above is the obvious generalization of [1], Theorem 13.6, p. 124, and its proof will be omitted.

As in the case of residuated mappings, there is a strong tie between the notions of range-closed and modularity. A further discussion of this topic will be covered in a later paper.

3. <u>Annihilator Properties of Range-Residuated Mappings</u>. In this section, it will be assumed that we are working in a fixed bounded poset $P$. Recall that $RR(P)$ is a semigroup with identity element $1$ and left zero elements $\{\kappa_x : x \in P\}$. The left zero element $\kappa_0$ will be of special interest. For $\phi \in RR(P)$, we define the <u>right annihilator</u> of $\phi$ by the rule

$$R(\phi) = \{\psi : \phi\psi = \kappa_{\phi(0)}\} ;$$

similarly, the left annihilator of $\phi$ is defined by

$$L(\phi) = \{\psi : \psi\phi = \kappa_{\phi(0)}\}.$$

We shall make strong use of the fact that

(5) $\phi\psi = \kappa_{\phi(0)} \Longleftrightarrow \psi(1) \leq \phi^+\phi(0)$.

The idea now is to relate order properties of the poset P to annihilator properties of the semigroup RR(P). To show that there is some hope in doing this, we let

$$\mathcal{R} = \{R(\phi):\phi \in RR(P)\}$$
$$\mathcal{L} = \{L(\phi):\phi \in RR(P)\}$$

with both sets partially ordered by set inclusion. We may then define mappings $F:\mathcal{R} \to P$, $G:\mathcal{L} \to P$ by the rules

$$F(R(\phi)) = \phi^+\phi(0)$$
$$G(L(\phi)) = \phi(1)$$

and note that $F$ *is an isomorphism of* $\mathcal{R}$ *onto* $P$, *and* $G$ *is a dual isomorphism of* $\mathcal{L}$ onto $P$. To see this, note first that if $R(\phi) \subseteq R(\alpha)$, then

$$\phi\kappa_{\phi^+\phi(0)} = \kappa_{\phi(0)} \Longrightarrow \alpha\kappa_{\phi^+\phi(0)} = \kappa_{\alpha(0)}$$

so that by (5), $\phi^+\phi(0) \leq \alpha^+\alpha(0)$. If conversely, $\phi^+\phi(0) \leq \alpha^+\alpha(0)$, then $\phi\psi = \kappa_{\phi(0)} \Longrightarrow \psi(1) \leq \phi^+\phi(0) \leq \alpha^+\alpha(0) \Longrightarrow \alpha\psi = \kappa_{\alpha(0)}$. So $R(\phi) \subseteq R(\alpha)$. We would be done if we could show $F$ to be onto. But this follows from the observation that if $\beta_x$ is defined by $\beta_x(p) = 0$ if $p \leq x$ and $1$ otherwise, then $\beta_x$ is residuated with $\beta_x^+\beta_x(0) = x$. A similar argument works for $G$. We now have

THEOREM 12. *Let P be a bounded poset. Then:*

(1) *P is a meet semilattice if and only if the right annihilator of each element of RR(P) is a principal right ideal generated by an idempotent.*

(2) P <u>is a join semilattice if and only if the left annihilator of each element of</u> RR(P) <u>is a principal left ideal generated by an idempotent</u>.

<u>Proof</u>: (1) Assume P to be a meet semilattice. Then for $p \in P$, we may define $\theta_p$ by the rule $\theta_p(x) = x$ $(x \leq p)$ and $p$ otherwise. Noting that $\theta_p$ is a range-closed idempotent residuated mapping, it follows from (5) that $\phi\psi = \kappa_{\phi(0)} \Longleftrightarrow \psi = \theta_{\phi+\phi(0)}\psi$. The converse follows from Theorem 4.

(2) If P is a join semilattice, then by (5), $\psi\phi = \kappa_{\phi(0)} \Longleftrightarrow \psi = \psi\tau_{\phi(1)}$. The converse follows from Theorem 5.

4. <u>Baer</u> LZ-<u>semigroups</u>. Let S be a semigroup with a two-sided zero element 0. For a given $x \in S$, define the left and right annihilators of $x$ by the rules

$$L(x) = \{y \in S : yx = 0\}$$
$$R(x) = \{y \in S : xy = 0\}.$$

To say that S is a Baer semigroup ([1], p. 104) is to say that for each $x \in S$ there correspond idempotents $e_x$, $f_x$ such that

$$L(x) = \{y \in S : y = yf_x\} = Sf_x$$
$$R(x) = \{y \in S : y = e_x y\} = e_x S.$$

An introduction to these semigroups is contained in [1], and an attempt is made there to relate properties of bounded posets to properties of suitable associated semigroups. For further details, the reader is referred to [1]. The link between Baer semigroups and lattices is made by means of certain residuated mappings. In order to develop a similar theory for

range-residuated mappings, one needs an analog of a Baer semigroup that only has a one-sided zero element. This we now proceed to introduce.

DEFINITION. A semigroup $S$ is said to be a Baer LZ-semigroup if

(1) $S$ has a distinguished left zero element $z$, and

(2) For each $x \in S$, there correspond idempotents $e_x$, $f_x$ such that

$$L(x) = \{y \in S : yx = yz\} = \{y \in S : y = yf_x\},$$
$$R(x) = \{w \in S : xw = xz\} = \{w \in S : w = e_x w\}.$$

Unless otherwise specified, $S$ will denote such a semigroup, and

$$\mathcal{L}(S) = \{L(x) : x \in S\}$$
$$\mathcal{R}(S) = \{R(x) : x \in S\}$$

with both $\mathcal{L}(S)$ and $\mathcal{R}(S)$ partially ordered by set inclusion. To say that a poset $P$ can be coordinatized by such an $S$ will be to say that $P$ is isomorphic to $\mathcal{R}(S)$. Note that if $z$ is a two-sided 0, then $S$ becomes a Baer semigroup in the sense of [1], p. 104. Note also that the left zero elements of $S$ correspond to the elements of the form $xz$ $(x \in S)$.

THEOREM 13. $S$ has a multiplicative identity.

Proof: Let $L(z) = Se$ and $R(z) = fS$ with $e, f$ idempotent. Then $R(z) = \{y \in S : zy = zz\} = S$ shows $f$ to be a right identity for $S$, while $L(z) = \{y \in S : yz = yz\} = S$ shows $e$ to be a left identity.

If we agree to let PRI(S), PLI(S) denote the set of principal right, left ideals of $S$ with both sets partially ordered by set inclusion, we also have

THEOREM 14. (1) *The mappings* $\hat{L}:PRI(S) \to PLI(S)$, $\hat{R}:PLI(S) \to PRI(S)$ *defined by* $\hat{L}(xS) = L(x)$, $\hat{R}(Sx) = R(x)$ *set up a galois connection in the sense of* [1], p. 18.

(2) $\hat{L} = \hat{L} \circ \hat{R} \circ \hat{L}$ *and* $\hat{R} = \hat{R} \circ \hat{L} \circ \hat{R}$.

(3) $xS \in \mathcal{R}(S) \iff xS = (\hat{R} \circ L)(x)$, *and*

$Sx \in \mathcal{L}(S) \iff Sx = (\hat{L} \circ R)(x)$.

(4) *The restriction of* $\hat{L}$ *to* $\mathcal{R}(S)$ *is a dual isomorphism of* $\mathcal{R}(S)$ *onto* $\mathcal{L}(S)$ *whose inverse is the restriction of* $\hat{R}$ *to* $\mathcal{L}(S)$.

Proof: In view of the similarity of this result to [1], Theorem 11.1, p. 95, we restrict our attention to the proof of (1).

If $xS \subseteq yS$, then $x = yw$ for some $w \in S$. Then $a \in L(y)$ implies $ay = az$, so $ax = ayw = azw = ax$. Thus

$$xS \subseteq yS \implies L(y) \subseteq L(x).$$

Similarly, if $Sx \subseteq Sy$, then $x = wy$, so $a \in R(y)$ implies $xa = wya = wyz = xz$, thereby putting $a \in R(x)$. In other words,

$$Sx \subseteq Sy \implies R(y) \subseteq R(x).$$

The fact that $a \in L(x)$ implies $ax = az$ also puts $x \in R(a)$, so $xS \subseteq (R \circ L)(xS)$; similarly, $Sx \subseteq (L \circ R)(Sx)$, thus completing the proof.

We shall frequently need

LEMMA 15. *If* $eS \in R(S)$ *with* $e = e^2$, *then* $z = ez$.

Proof: Let $eS = R(x)$. Since $z \in R(x)$, it follows that $z = ez$.

For $M$ a subset of $S$, we agree to let $R(M) = \{x : mx = mz \text{ for all } m \in M\}$ and note that if $R(M) = eS$ with $e = e^2$, then $eS = \wedge\{R(m) : m \in M\}$ in $\mathcal{R}(S)$. For each fixed $x \in S$, we define mappings $\phi_x, \eta_x : \mathcal{R} \to \mathcal{R}$ by the rules

$$\phi_x(eS) = (\hat{R} \circ L)(xe)$$
$$\eta_x(eS) = R(e^{\#}x)$$

where $Se^{\#} = L(e)$, and $e^{\#}$ is idempotent. The domain of $\eta_x$ is taken to be $\{eS \in \mathcal{R}(S) : \phi_x(zS) \subseteq eS\}$. From here on in, the elements $e,f,g,h$ (with or without superscripts) will, unless otherwise specified, denote idempotents. We agree further to let $\mathcal{R} = \mathcal{R}(S)$ and $\mathcal{L} = \mathcal{L}(S)$. We then have

THEOREM 16. *For each* $x \in S$, $\phi_x \in RR(\mathcal{R})$, *with* $\phi_x^+ = \eta_x$ .

Proof: We begin by showing $\phi_x$, $\eta_x$ to be well defined and isotone. Accordingly, let $eS \subseteq fS$ in $\mathcal{R}$. Then $e = fe$ and $y \in L(xf)$ implies

$$yxe = yxfe = yze = yz$$

thus showing $y \in L(xe)$. It follows that $\phi_x$ is well defined and isotone.

Now let $\phi_x(zS) \subseteq eS \subseteq fS$ in $R$, with $Se^{\#} = L(e)$ and $Sf^{\#} = L(f)$. Then $L(f) \subseteq L(e)$, so $f^{\#} = f^{\#}e^{\#}$. If $y \in R(e^{\#}x)$, then $e^{\#}xy = e^{\#}xz$, and then

$$f^{\#}xy = f^{\#}e^{\#}xy = f^{\#}e^{\#}xz = f^{\#}xz,$$

thus putting $y \in R(f^{\#}x)$. Consequently, $\eta_x$ is well defined and isotone.

Suppose now that $\phi_x(eS) \subseteq fS$ in $R$. Then $\phi_x(zS) \subseteq fS$, so $xz = fxz$, and $f^{\#}xz = f^{\#}fxz = f^{\#}z$. It follows that

$$f^{\#}xe = f^{\#}fxe = f^{\#}z = f^{\#}xz,$$

whence $eS \subseteq R(f^{\#}x)$. On the other hand, if $\phi_x(zS) \subseteq fS$, and $eS \in R(f^{\#}x)$, then

$$f^{\#}xe = f^{\#}xz = f^{\#}z$$

puts $xe$ in $R(Sf^{\#}) = (\hat{R} \circ \hat{L})(fS)$, so $\phi_x(eS) = (\hat{R} \circ L)(xe) \subseteq fS$. This shows that $\eta_x = \phi_x^{+}$, as claimed.

Actually as is seen by the next result, $L = R(S)$ is in fact a bounded lattice. The proof is similar to that of (1), Theorem 12.2, p. 107.

<u>LEMMA 17</u>. $L = R(S)$ <u>is a bounded lattice</u>.

<u>Proof</u>: Let $eS, fS \in L$ with $Se^{\#} = L(e)$, and $Sf^{\#} = L(f)$. If $gS = R(f^{\#}e)$, then

$$(f^{\#}e)(eg) = f^{\#}eg = f^{\#}ez$$

shows $eg \in R(f^{\#}e) = gS$, so $eg = geg$ and $eg$ is idempotent. Now let $x \in R(\{e^{\#},f^{\#}\})$. Then

$$e^{\#}x = e^{\#}z \Longrightarrow x = ex,$$

so

$$f^{\#}ex = f^{\#}x = f^{\#}z = f^{\#}ez$$

puts $x \in R(f^{\#}e) = gS$, and $x = gx = egx$.

If conversely, $x = egx$, then

$$e^{\#}x = e^{\#}egx = e^{\#}z$$
$$f^{\#}x = f^{\#}egx = f^{\#}ez = f^{\#}z$$

puts $x \in R(\{e^{\#},f^{\#}\})$. It is immediate that $eS \cap fS = egS \in L$, and this shows $L$ to be a meet semilattice.

In order to show that $L$ is a join semilattice, it suffices by Theorem 14 to show that $\mathcal{L}(S)$ is a meet semilattice. Accordingly, we let $Se, Sf \in \mathcal{L}(S)$ with $e'S = R(e)$, $f'S = R(f)$, and $Sg = L(ef')$. We shall show that $Sf \cap Se = Sg \cap Se = Sge$. Note first that

$$(ge)(ef') = gef' = gz.$$

By Lemma 15,

$$gez = gef'z = gz,$$

so $(ge)(ef') = gz = gez$, and $ge \in L(ef') = Sg$. It follows that $ge = geg$, so $ge$ is idempotent.

If $x \in L(\{e',f'\})$ then $xe' = xz$, so $x = xe$. It follows that $xef' = xf' = xz$, and $x = xg$. Consequently, $x = xg = xge$. On the other hand, if $x = xge$, then

$$xe' = xgee' = xgez = xz,$$

so $x \in L(e')$. Also, a second application of Lemma 15 produces

$$xf' = xgef' = xgz = xgez = xz$$

thus showing that $x \in L(f')$.

An immediate consequence of Theorem 12 and Lemma 17 is

THEOREM 18. *For a bounded poset* $P$, *the following conditions are equivalent*:

(1) $P$ *is a lattice*.

(2) $RR(P)$ *is a Baer* LZ-*semigroup*.

(3) $P$ *can be coordinatized by a Baer* LZ-*semigroup*.

The question of what it means for the mapping $x \to \phi_x$ to be a semigroup homomorphism of $S$ into $RR(\mathcal{R}(S))$ is settled by

THEOREM 19. *Let* $S$ *be a Baer* LZ-*semigroup, and* $L = \mathcal{R}(S)$. *The following conditions are then equivalent*:

(1) *The mapping* $x \to \phi_x$ *is a semigroup homomorphism of* $S$ *into* $RR(L)$.

(2) $\phi_x(zS) \leq \phi_{xy}(zS)$ *for every* $x,y$ *in* $S$.

(3) $a \in L(xyz) \Longrightarrow ax \in L(yz)$ *for all* $x,y$ *in* $S$.

*Proof*: (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Let $a \in L(xyz)$. By hypothesis, $\phi_x(zS) \leq \phi_{xy}(zS)$, so $L(xyz) \subseteq L(xz)$. Thus $a \in L(xyz) \Rightarrow a \in L(xz)$, whence $axz = az$. But then $axyz = az = axz$ puts $ax \in L(yz)$, as claimed.

(3) ==> (1). For $eS \in L$, $\phi_x\phi_y(eS) = (\hat{R} \circ L)(xg)$, where $gS = (\hat{R} \circ L)(ye)$, and $\phi_{xy}(eS) = (\hat{R} \circ L)(xye)$. We would be done if we could show that $L(xg) = L(xye)$. To see this, note that

$$a \in L(xg) \Longrightarrow ax \in L(g) = L(ye).$$

Thus

$$az = axz = axg = axye,$$

and this puts $a \in L(xye)$. The reverse inclusion is established in a similar manner.

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